Eur. Phys. J. B **28**, 71–74 (2002)
DOI: 10.1140/epjb/e2002-00201-3

THE EUROPEAN PHYSICAL JOURNAL B

Breakdown of waves described by exact solutions of the Thomas-Fermi model

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Received 22 March 2002
Published online 9 July 2002 – © EDP Sciences, Società Italiana di Fisica, Springer-Verlag 2002

Abstract. The dynamics of large localized repulsive clouds is examined by means of exact non-stationary solutions of the one-dimensional Thomas-Fermi model. The nonlinear flattening of the cloud peak, the wave breakdown at the cloud peripheries, and the condensate velocity distributions are thus described. Our solutions, which can contain an arbitrary amount of free parameters, show the nonlinear evolution of an arbitrary initial wave form. A unique procedure for analyzing these solutions is presented. The difference between our breakdown matter wave solutions and the well known Riemann shock waves is stressed.

PACS. 03.75.Fi Phase coherent atomic ensembles; quantum condensation phenomena 04. General relativity and gravitation – 03.65.Ge Solutions of wave equations: bound states – 02.70.Ns Molecular dynamics and particle methods

1 Introduction

After the discovery [1–3] of Bose-Einstein condensate (BEC) in trapped clouds of dilute atomic gases (alkali atoms), there has been a great deal of interest [4,5] in investigating the nonlinear dynamics of matter waves. The amplitudes and phases of the latter are usually described by the time dependent Gross-Pitaevskii equation [6] or a nonlinear Schrödinger equation (NLSE). Significant progress has recently been made in quasi-one dimensional condensate confinement, such as for example in atomic waveguides [7], cigar-shaped BEC and toroidal traps [8,9]. The condensate is then strongly elongated (in the z-direction). This means that the condensate remains essentially in its ground state in a plane orthogonal to the wave direction, and that its wave function is required to vanish at the confining walls. Only the longitudinal excitations along the z-directions are then of interest. This approach leads to the theory for one-dimensional stationary nonlinear self-consistent distributions for the density and velocity in BEC clouds including periodical cnoidal waves with both repulsive [10] and attractive [11] nonlinearities, as well as observations of dark BEC solitons [12]. Contrary to previous approaches, the present paper is devoted to the *non-stationary* reshaping of one-dimensional solitary distributions of BECs with repulsive nonlinearities. In Section 2, we show how the NLSE is transformed to a nonlinear Thomas-Fermi (TF) model which describes the evolution of the matter waves. Exact analytical solutions of the nonlinear TF model equations are presented. Also given are conditions under which the matter waves break. Section 3 contains a brief summary of our work.

2 Formulation

We study phenomena that include the damping and flattening of BEC distributions, the steepening of their density profiles at the periphery, and the formation of the velocity profile. These effects are described by the time dependent NLSE

$$i\hbar\partial_t\psi + \frac{\hbar^2}{2m}\partial_z^2\psi - \kappa g|\psi|^2\psi = 0, \tag{1}$$

where $\hbar$ is the Planck constant, the coupling constant g is connected with the scattering length $a(>0)$ through [13]

$$g = \frac{4\pi\hbar^2 a}{m}, \tag{2}$$

and m is the atom mass. The dimensionless coefficient κ has appeared due to integration of the wave function over the transverse coordinates, similar to what is done in nonlinear fiber optics [14]. Its numerical value is determined by the type of confinement [15].

We write the wavefunction in the form [16]

$$\psi = [n_0 W(t,z)]^{1/2}\exp[iS(t,z)], \tag{3}$$

where n_0 is the maximum value of the BEC density and the dimensionless function W describes the density distribution in time and space. Both W and S are real.

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Substituting (3) into (1), separating into real and imaginary parts, and differentiating the real part with respect to z, we obtain

$$\partial_t v + v\partial_z v + \frac{g\kappa n_0}{m}\partial_z W = \frac{\hbar^2}{2m^2}\partial_z\left(\frac{1}{\sqrt{W}}\partial_z^2\sqrt{W}\right), \quad (4)$$

and

$$\partial_t W + \partial_z(Wv) = 0, \quad (5)$$

where $v(t,z)$ is the condensate velocity

$$v = \frac{\hbar}{m}\partial_z S. \quad (6)$$

Comparing the terms in (4) with each other, one can see that the right-hand side in (4) can be neglected if the length L of the elongated BEC cloud is larger than the healing length $l = (8\pi a n_0)^{-1/2}$. Usually, $l \approx 100$ nm and $L \approx 10^4$ nm, *i.e.* the condition $L \gg l$ is easily satisfied. Thus, omitting the right-hand side in (4) we obtain the Thomas-Fermi (TF) model describing large condensates [9]. Introducing the normalized variables $\eta = z/L$, $\tau = t\sqrt{\kappa}v_s/L$, and $u = v/\sqrt{\kappa}v_s$, where v_s is the Bogoliubov sound speed

$$v_s = \left(\frac{gn_0}{m}\right)^{1/2} \quad (7)$$

we can rewrite the TF limit of equations (4, 5) in the dimensionless form

$$\partial_\tau u + u\partial_\eta u + \partial_\eta W = 0, \quad (8)$$

and

$$\partial_\tau W + \partial_\eta\,(Wu) = 0. \quad (9)$$

The TF model equations (8, 9) describe the large-scale dynamics of matter waves. In the following, we shall present a broad family of exact non-stationary solutions of the TF model. They will then be used to analyze the self-reshaping of solitary matter waves, *i.e.* to describe the flattening of the wave peak and the steepening of the density distribution at the solitary wave periphery leading to the breakdown of the matter waves.

Exact analytical solutions of the one-dimensional TF model are found as follows. The evolution of the spatio-temporal distributions of the density $W(\tau,\eta)$ and the velocity $u(\tau,\eta)$ can be obtained from equations (8, 9) by means of the following steps:

a) We exchange the independent variables τ and η and the function W and u, *i.e.* we consider the quantities W and u as independent variables and τ and η as the new unknown functions $\tau = \tau(W,u)$ and $\eta = \eta(W,u)$. The TF system (8) and (9) is then transformed to

$$\frac{\partial\eta}{\partial W} - u\frac{\partial\tau}{\partial W} + \frac{\partial\tau}{\partial u} = 0 \quad (10)$$

and

$$\frac{\partial\eta}{\partial u} + W\frac{\partial\tau}{\partial W} - u\frac{\partial\tau}{\partial u} = 0. \quad (11)$$

This transformation has thus, without any approximations, reduced the nonlinear TF system (8) and (9) to a system of linear equations (10, 11).

b) We next reduce the system of two equations (10, 11) to one equation by introducing a generating function $F(W,u)$, and using the hodograph transform [17]

$$\tau = \partial F/\partial W \qquad \text{and} \qquad \eta = u\tau - \partial F/\partial u. \quad (12)$$

Equation (10) then reduces to an identity, whereas equation (11) yields

$$W\frac{\partial^2 F}{\partial W^2} + \frac{\partial F}{\partial W} - \frac{\partial^2 F}{\partial u^2} = 0. \quad (13)$$

Contrary to the traditional approach [17], it is now convenient to introduce the coordinates ξ and ρ instead of W and u from

$$W = (1-\xi^2)(1-\rho^2) \qquad \text{and} \qquad u = -2\xi\rho, \quad (14)$$

where $-1 \le \xi \le 0$ and $-1 \le \rho \le 1$. We can then transform equation (13) for the function $F(\xi,\rho)$ to the nice equation

$$\frac{\partial}{\partial\xi}\left[(1-\xi^2)\frac{\partial F}{\partial\xi}\right] = \frac{\partial}{\partial\rho}\left[(1-\rho^2)\frac{\partial F}{\partial\rho}\right]\cdot \quad (15)$$

Equation (15), which has been derived from equations (8, 9) without any approximation, contains thus all information about the dynamics of a BEC distribution within the Thomas-Fermi model.

Equation (15) is a hyperbolic equation. Instead of solving it in the traditional way by means of characteristics, we shall express the solution of (15) in terms of its orthogonal eigenfunctions. The advantage of this method will be obvious below. We then use the method of separation of variables, writing

$$F(\xi,\rho) = f_1(\xi)f_2(\rho), \quad (16)$$

which leads to the equation

$$(1-\xi^2)\frac{\partial^2 f_1}{\partial\xi^2} - 2\xi\frac{\partial f_1}{\partial\xi} + j\ (j+1)f_1 = 0, \quad (17)$$

where $j = 0,1,2,...$, and where the equation for f_2 is obtained by replacing ξ by ρ and f_1 by f_2. The eigenfunctions of (17) are the Legendre functions of the first (P_j) and second (Q_j) kind.

The solution of equation (15) can thus be written as

$$F = \sum_{j=0}^{\infty} Q_j(\rho)\left[A_jQ_j(\xi) + B_jP_j(\xi)\right] + P_j(\rho)\left[C_jQ_j(\xi) + D_jP_j(\xi)\right]. \quad (18)$$

where the unknown coefficients A_j, B_j, C_j and D_j have to be determined from the initial conditions, describing the BEC cloud at $t = 0$.

Rewriting (12) in the new coordinates ξ and ρ we have

$$\tau = \frac{1}{2\ (\rho^2-\xi^2)}\left(\xi\frac{\partial F}{\partial\xi} - \rho\frac{\partial F}{\partial\rho}\right) \quad (19)$$

and

$$\eta = \frac{1}{2(\rho^2 - \xi^2)} \left[\rho(1 - 3\xi^2)\frac{\partial F}{\partial \xi} - \xi(1 - 3\rho^2)\frac{\partial F}{\partial \rho}\right] \cdot \quad (20)$$

We have thus found an exact analytical solution of the nonlinear Thomas-Fermi model equations (8, 9). In the Appendix, we have presented formulas for differentiation in (ξ, ρ) space with respect to W and u, as well as given the expressions for the eigenfunctions P_j and Q_j. To describe the dynamics of any BEC distribution by means of (19) and (20) we have to use the initial condition.

c) We now consider the simple case where the BEC cloud at $t = 0$ is supposed to be immobile, *i.e.* $u(0, \eta) = 0$. We denote $W(0, \eta)$ by W_0. For the functions $\tau(\xi, \rho)$ and $\eta(\xi, \rho)$ we then write $\tau(0, \rho) = 0$ and $\eta(0, \rho) = \eta_0$. Using (19) and (20) we have $\partial F/\partial\rho|_{\xi=0} = 0$ and $\partial F/\partial\xi|_{\xi=0} = 2\rho\eta_0$. Substituting the derivatives of the function (18) into the expressions (12) we obtain a closed system of algebraic equations for calculating the unknown coefficients A_j, B_j, C_j and D_j. This approach is valid for any initial waveform that can be written as a sum of Legendre functions. One can easily consider the dynamics of a multitude of initial profiles.

A typical example of the nonlinear dynamics of the matter waves that are described by such solutions will be considered below. It is, however, first worthwhile to mention that the standard solutions of the TF system using parabolic and hyperbolic-secant profiles of $W(0, \eta)$ are limiting cases of the approach developed above. Thus, the parabolic profile $W_0 = 1 - \eta^2$ ($\eta^2 \leq 1$) can be written as $\eta_0 = P_1(\rho)$, and the profile $W_0 = \mathrm{ch}^{-2}(\eta)$ relates to $\eta_0 = Q_0(\rho)$. These profiles, which have no free parameters, are represented by the first eigenfunctions of equation (17).

Contrary to this, we can consider more general initial profiles of $\eta_0(\rho)$ built from the harmonics $P_j(\rho)$ and $Q_j(\rho)$ and containing an arbitrary amount of free parameters. As an example, we consider a simple bell-like profile containing one free parameter M (< -1), *i.e.*

$$\eta_0 = \frac{1 - \rho^2}{2M} \ln \frac{1 + \rho}{1 - \rho} + \rho. \quad (21)$$

Unlike the ch^{-2} profile, the profile (21) has a finite width between the points $\eta_0 = \pm 1$, and unlike the parabolic profile, the edges of (21) are smooth, *i.e.* $W_0|_{\eta=1} = W_0|_{\eta=-1} = 0$ and $(\partial W_0/\partial\eta)_{\eta=1} = (\partial W_0/\partial\eta)_{\eta=-1} = 0$. The parameter M determines the half-width of the distribution whereas the maximum point ($W = 1$), the edges $[W(1) = W(-1) = 0]$ and the derivatives at these points are fixed. Expressing the product $2\rho\eta_0(\rho)$, where η_0 is given by (21), in terms of the eigenfunctions P_j and Q_j we have

$$\rho \left[\frac{1 - \rho^2}{M} \ln \frac{1 + \rho}{1 - \rho} + 2\rho\right] = -\frac{4}{5M} Q_3(\rho) + \frac{4}{5M} Q_1(\rho) + \frac{4}{3}\left(1 - \frac{1}{M}\right) P_2(\rho) + \frac{2}{3}\left(1 + \frac{1}{M}\right) P_0(\rho), \quad (22)$$

which can be substituted into the initial conditions to find the coefficients in (18). The result is

$$C_0 = \frac{2}{3}\left(1 + \frac{1}{M}\right), \quad B_1 = \frac{4}{5M}, \quad C_1 = \frac{2}{3}\left(\frac{1}{M} - 1\right) \quad \text{and} \quad B_3 = \frac{8}{15M} \quad (23)$$

whereas all the other coefficients in (18) are equal to zero. Substituting the function F in (19) and (21) we then finally obtain the expressions describing the self-reshaping of the BEC distribution (21). Thus

$$\tau = \frac{1}{2}\left(\frac{1}{M} - 1\right)\left[\frac{1}{2} \ln \frac{1 + \xi}{1 - \xi} + \frac{\xi}{1 - \xi^2}\right] + \frac{\xi}{M}\left[\rho \ln \frac{1 + \rho}{1 - \rho} - 2 - \frac{2}{3} \frac{\xi^2}{(1 - \xi^2)(1 - \rho^2)}\right] \quad (24)$$

and

$$\eta = \frac{1}{M}\left[\frac{1 - \xi^2 - \rho^2 - \xi^2\rho^2}{2} \ln \frac{1 + \rho}{1 - \rho} + \frac{\rho\xi^4(1 + 3\rho^2)}{3(1 - \xi^2)(1 - \rho^2)}\right] + \frac{\rho}{1 - \xi^2} \cdot \quad (25)$$

One can see that the initial profile (21) is obtained from (24) and (25) in the case $\xi = 0$ (*i.e.* $t = 0$).

We cannot rewrite the relations $\tau = \tau(\xi, \rho)$ and $\eta = \eta(\xi, \rho)$, as defined by (24) and (25), in the alternative form $\xi = \xi(\tau, \eta)$ and $\rho = \rho(\tau, \eta)$. However, by means of (24) and (25) one can easily deduce that the spatio-temporal evolution of the initial profile (21) leads to the formation of a non-stationary velocity distribution inside the BEC cloud, a damping of the peak and a flattening of the density profile, and a steepening of the periphery region of the profile, *i.e.* to wave breakdown. Determining the bend point values ξ_b and η_b from $\partial W/\partial\eta|_\tau = 0$ and $\partial^2 W/\partial\eta^2|_\tau = 0$, and substituting these values into (24) and (25) we find the coordinates for breakdown $\tau_b = \tau(\xi_b, \rho_b)$ and $\eta_b = \eta(\xi_b, \rho_b)$. When $M = -1.25$ we have $\tau_b = 0.52$ and $\eta_b = 0.71$. The peak density and velocity at the wave pulse maximum are $W = 0.51$ and $u = 0$. The same quantities at the breakdown point are $W_b = 0.38$ and $u_b = 0.59$. The bell-like waveform is transformed due to this self-reshaping, leading to a distribution with a broad flattened top and steep periphery regions. We note that (24) and (25) are only valid before breakdown. Another result of this solitary wave reshaping is the creation of a new minimum and subsequent self-splitting of the initially bell-like waveform.

In a similar way we can examine the nonlinear dynamics of more complicated profiles where the initial waveform $\eta_0 = \eta(\rho)$ contains more Legendre terms.

It should be stressed that by writing the solution of TF model in terms of the eigenfunctions, we are able to examine the formation of shock waves where the density and velocity are not limited by a particular relation $W = W(u)$. Thus our waves generally differ from the Riemann waves [17]. However, the TF model admits, of course, also

Riemann waves as particular solutions. Considering the equations of characteristics of equation (13) we thus find the Riemann wave

$$W = \left(\frac{u}{2} + C\right)^2, \tag{26}$$

where C is a constant. An analysis of such a Riemann wave, that for example can be formed in a magnetized plasma, was presented in reference [18]. That analysis can thus be generalized by using the eigenfunctions of (17) to consider the formation and breakdown of waves that are different from Riemann waves.

3 Summary

To summarize, we have examined the non-stationary reshaping of elongated BEC clouds that are described by the Thomas-Fermi model. A wide family of exact analytical solutions of this model has been found for various initial solitary matter waveforms. They are expressed in terms of simple well known elementary functions. By introducing a curvilinear coordinate system it has been possible to describe the spatiotemporal dynamics leading to a decrease and broadening of the peak of the solitary wave as well as to a steepening of its periphery parts. These processes, which also cause a velocity distribution in the initially immobile cloud, lead to breakdown of the matter waves. A method to calculate the breakdown coordinates is presented. Our analysis can also be used to construct other classes of nonlinear solutions, including self-splitted waveforms and non-Riemann shocks.

LS and PKS acknowledge useful discussions with Eric Cornell during his recent visit to Umeå University. AS thanks Tito Arecchi and Serge Haroche for valuable comments. This work was partially supported by the Swedish Natural Science Research Council and the Royal Swedish Academy of Sciences.

Appendix

It follows from (14) that the formulas for differentiation in (ξ, ρ) space with respect to W and u are

$$\frac{\partial}{\partial W} = \frac{1}{2(\rho^2 - \xi^2)} \left(\xi \frac{\partial}{\partial \xi} - \rho \frac{\partial}{\partial \rho}\right)$$

and

$$\frac{\partial}{\partial u} = \frac{1}{2(\rho^2 - \xi^2)} \left[\xi\ (1-\rho^2)\frac{\partial}{\partial \rho} - \rho\ (1-\xi^2)\frac{\partial}{\partial \xi}\right].$$

Furthermore, the eigenfunctions P_j and Q_j used in the present paper are

$$P_0(\xi) = 1, \quad P_1(\xi) = \xi, \quad P_2(\xi) = \frac{(3\xi^2 - 1)}{2}, \quad P_3(\xi) = \frac{(5\xi^3 - 3\xi)}{2},$$

$$Q_0(\xi) = \frac{1}{2}\ln\left(\frac{1+\xi}{1-\xi}\right), Q_1(\xi) = \frac{\xi}{2}\ln\left(\frac{1+\xi}{1-\xi}\right) - 1,$$

$$Q_2(\xi) = \frac{(3\xi^2-1)}{4}\ln\left(\frac{1+\xi}{1-\xi}\right) - \frac{3\xi}{2},$$

$$\text{and} \quad Q_3(\xi) = \frac{(5\xi^3 - 3\xi)}{4}\ln\left(\frac{1+\xi}{1-\xi}\right) - \frac{5\xi^2}{2} + \frac{2}{3}.$$

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